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ON THE MOTIVE OF AN ABELIAN SCHEME WITH NON-TRIVIAL ENDOMORPHISMS

by Ben Moonen

Abstract. Let X be an abelian scheme over a base variety S and let $D = \mathrm{End}(X/S) \otimes \mathbb{Q}$ be its endomorphism algebra. We prove that the relative Chow motive of X has a natural decomposition as a direct sum of motives $R^{(\alpha)}$ where α runs over an explicitly determined finite set. To each α corresponds an irreducible representation ρ_α of the group $D^{\mathrm{opp},*}$ and the motivic decomposition is such that $R^{(\alpha)}$, as a functor on the category of relative Chow motives, is a sum of copies of ρ_α. In particular $\mathrm{CH}(R^{(\alpha)})$, as a representation of $D^{\mathrm{opp},*}$, is a sum of copies of ρ_α. Our decomposition refines the motivic decomposition of Deninger and Murre, as well as Beauville's decomposition of the Chow group.

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Introduction

As an application of Fourier theory, Beauville proved in [2] that the Chow ring (with $\mathbb{Q}$-coefficients) of a g-dimensional abelian variety X has a bigrading $\mathrm{CH}(X) = \oplus_{j,s} \mathrm{CH}^j_{(s)}(X)$, where the upper grading is given by the codimension of cycles and $[m]_X^*$ acts on $\mathrm{CH}^j_{(s)}(X)$ as multiplication by m^{2j-s}. As shown by Deninger and Murre in [4], this decomposition in fact comes from a natural decomposition $R(X) = \oplus_{i=0}^{2g} R^i(X)$ of the Chow motive of X; we have $\mathrm{CH}^j_{(s)}(X) = \mathrm{CH}^j\bigl(R^{2j-s}(X)\bigr)$. The results of Deninger and Murre are valid, more generally, for abelian schemes $X \to S$ over a smooth quasi-projective base variety over a field.

One way to state Beauville's result is by saying that $\mathbb{Q}^*$ acts on the Chow ring (letting $m/n \in \mathbb{Q}^*$ act as $[m]_X^* \circ [n]_X^{*,-1}$), and that the only characters that occur in this representation are the characters $q \mapsto q^i$ for $i \in \{0, 1, \ldots, 2g\}$. The main purpose of this paper is to explain how this can be refined in the presence of non-trivial endomorphisms.

To describe our main result, consider an abelian scheme $X \to S$ of relative dimension g that is isogenous to a power of a simple abelian scheme. (This is the essential case, to which the general case is reduced; see (5.5).) The endomorphism algebra $D = \mathrm{End}(X/S) \otimes \mathbb{Q}$ is then a simple algebra with center a number field K. Let Γ denote the Galois group of the normal closure of K over $\mathbb{Q}$. The group $D^{\mathrm{opp},*}$ acts on $\mathrm{CH}(X)$ and on the motives $R^i(X/S)$, which are objects of the category $\mathbf{M}^0(S)$ of relative Chow motives over S. This induces the structure of a $D^{\mathrm{opp},*}$-representation on $\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M, R(X/S)\bigr)$, for any relative Chow motive M.

Let G be $D^{\mathrm{opp},*}$, viewed as a reductive group over $\mathbb{Q}$. The irreducible representations of G over $\mathbb{Q}$ are indexed by the Γ-orbits in a space $\mathbb{X}^+$ of highest weight vectors. Write ρ_α for the irreducible representation of $D^{\mathrm{opp},*} = G(\mathbb{Q})$ corresponding to $\alpha \in \mathbb{X}^+/\Gamma$. There is a natural "weight function" $\|\ \| \colon \mathbb{X}^+/\Gamma \to \mathbb{Z}$ that sends a class α to the degree of the restriction of ρ_α to the subgroup $\mathbb{G}_m \subset G$ of homotheties. Further, we consider an explicit finite subset $\mathbb{X}^{\mathrm{adm}}/\Gamma \subset \mathbb{X}^+/\Gamma$ of "admissible" elements; see (4.2) for the definition.

Our main results are Theorems (4.3) and (5.1) in the text. The content of these results is

that there is a unique motivic decomposition

$$R(X/S) = \bigoplus_{\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma} R^{(\alpha)}(X/S)$$

that is stable under the action of $D^{\mathrm{opp},*}$ and has the property that for any motive M the $D^{\mathrm{opp},*}$-representation $\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M, R^{(\alpha)}(X/S)\bigr)$ is isomorphic to a sum of copies of the irreducible representation ρ_α. In particular, the Chow group $\mathrm{CH}\bigl(R^{(\alpha)}(X/S)\bigr)$ is a sum of copies of ρ_α as a representation of $D^{\mathrm{opp},*}$. For $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ we have $0 \leqslant \|\alpha\| \leqslant 2g$ and $R^i(X/S)$ is the direct sum of the motives $R^{(\alpha)}(X/S)$ with $\|\alpha\| = i$.

Further we describe an involution $\alpha \mapsto \alpha^\star$ on the set $\mathbb{X}^{\mathrm{adm}}/\Gamma$, with $\|\alpha^\star\| = 2g - \|\alpha\|$, and we obtain a motivic Poincaré duality isomorphism $R^{(\alpha)}(X/S)^\vee \xrightarrow{\sim} R^{(\alpha^\star)}(X/S)\bigl(g\bigr)$. Finally, if $X^\dagger/S$ is the dual abelian scheme, we have a motivic Fourier duality $\mathscr{F}\colon R^i(X/S) \xrightarrow{\sim} R^{2g-i}(X^\dagger/S)\bigl(g-i\bigr)$ and we prove that this $\mathscr{F}$ is a sum of isomorphisms $R^{(\alpha)}(X/S) \xrightarrow{\sim} R^{(\alpha^\star)}(X^\dagger/S)\bigl(g-i\bigr)$, for $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ with $\|\alpha\| = i$.

The proof of our results relies on the fact that the group $D^{\mathrm{opp},*}$ acts on $\mathrm{CH}\bigl(R^i(X/S)\bigr)$ through a representation that is polynomial of degree i, by which we mean that all matrix coefficients that occur in this representation are homogeneous polynomial functions of degree i on D. In Section 1 we discuss the classification of such representations. The proof that the representation on $\mathrm{CH}\bigl(R^i(X/S)\bigr)$ is indeed of this kind reduces, via Künnemann's isomorphism $R^i(X/S) \cong \wedge^i R^1(X/S)$, to the case $i = 1$, in which case it is the unsurprising assertion that the natural map $D^{\mathrm{opp}} \to \mathrm{End}\bigl(R^1(X/S)\bigr)$ given by $f \mapsto f^*$ is a homomorphism of $\mathbb{Q}$-algebras. In Section 4 we study the decomposition of $\mathrm{CH}(X)$ and by bootstrapping we obtain from this in Section 5 a motivic decomposition.

Conventions. — Throughout, Chow groups are taken with $\mathbb{Q}$-coefficients. All group actions we consider are left actions.

1. Some inputs from representation theory

(1.1) In this section we consider a simple algebra B of finite dimension over a field k of characteristic 0. Let K be the center of B, let $[K:k] = n$ and let $d = \dim_K(B)^{1/2}$.

Let $\bar{k}$ be an algebraic closure of k and let $\Sigma(K)$ denote the set of k-algebra homomorphisms $K \to \bar{k}$. Let $\tilde{K}$ denote the normal closure of K inside $\bar{k}$, and write $\Gamma = \mathrm{Gal}(\tilde{K}/k)$. The natural action of $\mathrm{Gal}(\bar{k}/k)$ on $\Sigma(K)$ factors through an action of Γ.

(1.2) Let H be the reductive group over K with $H(R) = (B \otimes_K R)^*$ for any commutative K-algebra R. Let $\bigl(\mathbb{X}(H), \Phi, \mathbb{X}^\vee(H), \Phi^\vee, \Delta\bigr)$ be the based root datum of H. We need to recall the definition of $\mathbb{X}(H)$; see for instance [10], Section 1.2, for further details. Consider pairs (T, Q) consisting of a maximal torus $T \subset H_{\bar{K}}$ and a Borel subgroup $Q \subset H_{\bar{K}}$ containing T. Given such a pair, let $\mathbb{X}_{(T,Q)}$ denote the character group of T. If (T', Q') is another pair, there exists an element $h \in H(\bar{K})$ such that $hTh^{-1} = T'$ and $hQh^{-1} = Q'$. The induced isomorphism $\mathbb{X}_{(T',Q')} \xrightarrow{\sim} \mathbb{X}_{(T,Q)}$ is independent of the choice of h and $\mathbb{X}(H)$ is defined as the projective limit of the groups $\mathbb{X}_{(T,Q)}$. For any pair (T, Q) the natural map $\mathbb{X}(H) \to \mathbb{X}_{(T,Q)}$ is an isomorphism.

There is a natural choice for an ordered $\mathbb{Z}$-basis $\{e_1, \ldots, e_d\}$ of $\mathbb{X}(H)$, obtained in the following way. Choose an isomorphism of $\bar{K}$-algebras a: $B \otimes_K \bar{K} \xrightarrow{\sim} M_d(\bar{K})$; this induces an isomorphism α: $H_{\bar{K}} \xrightarrow{\sim} \mathrm{GL}_{d,\bar{K}}$. Let $T \subset Q \subset H_{\bar{K}}$ be the maximal torus and Borel subgroup such that $\alpha(T)$ is the diagonal torus and $\alpha(Q)$ is the upper triangular Borel. Let ϵ'_j: $\alpha(T) \to \mathbb{G}_{m,\bar{K}}$ be the character that sends a diagonal matrix with entries $(c_1, \ldots, c_d)$ to c_j, and define $\epsilon_j \in \mathbb{X}_{(T,Q)}$ by $\epsilon_j = \epsilon'_j \circ \alpha$. Then $\{\epsilon_1, \ldots, \epsilon_d\}$ is an ordered $\mathbb{Z}$-basis of $\mathbb{X}_{(T,Q)}$. Now define $\{e_1, \ldots, e_d\}$ to be the ordered $\mathbb{Z}$-basis of $\mathbb{X}(H)$ such that $e_j \mapsto \epsilon_j$ under the isomorphism $\mathbb{X}(H) \xrightarrow{\sim} \mathbb{X}_{(T,Q)}$. It follows from the Skolem-Noether theorem and the definition of $\mathbb{X}(H)$ that the ordered basis thus obtained does not depend on the choice of the isomorphism a. Further it is clear from the construction that the roots are the vectors $e_i - e_j$ for $i \neq j$, and that the basis of positive roots is given by $\Delta = \{e_i - e_{i+1} \mid i = 1, \ldots, d-1\}$.

(1.3) The group H is an inner form of GL_d; hence the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts trivially on the root datum of H. By [9], Thm. 7.2, we have a bijective correspondence between the set of irreducible finite-dimensional representations of H over K and the set $\mathbb{X}(H)^+$ of dominant weights.

With respect to the ordered basis $\{e_1, \ldots, e_d\}$ as in (1.2), the dominant weights are the vectors $\lambda_1 e_1 + \cdots + \lambda_d e_d$ for $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d$. This gives an identification of $\mathbb{X}(H)^+$ with the set

(1.3.1) $$\Lambda^+ = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d\}.$$

For $\lambda \in \Lambda^+ = \mathbb{X}(H)^+$, let ψ_λ be the corresponding irreducible representation of H over K.

If ϕ_λ is the irreducible representation of GL_d with highest weight given by λ, the representation ψ_λ is a K-form of the representation $\phi_\lambda^{\oplus d(\lambda)}$ for some integer $d(\lambda)$ that divides d. For later use, let us also recall that if $\lambda_d \geqslant 0$, the representation ϕ_λ is the one obtained from the standard representation of GL_d applying the Schur functor $\mathbb{S}_\lambda$. In the general case, without the assumption that $\lambda_d \geqslant 0$, we take an integer m with $\lambda_d + m \geqslant 0$; then $\phi_\lambda = \phi_{(\lambda_1+m,\ldots,\lambda_d+m)} \otimes \det^{-m}$. See for instance [5], Section 15.5.

(1.4) Next we consider the reductive group $G = \mathrm{Res}_{K/k} H$ over k. If R is a commutative k-algebra, $G(R) = (B \otimes_k R)^*$. The set $\mathbb{X}(G)^+$ of dominant weights of $G_{\bar{k}}$ is given by $\mathbb{X}(G)^+ = \oplus_{\sigma \in \Sigma(K)} \mathbb{X}(H)^+$. Via the identification $\mathbb{X}(H)^+ = \Lambda^+$ of (1.3), we obtain an identification of $\mathbb{X}(G)^+$ with the set

$$\mathbb{X}^+ = \bigoplus_{\sigma \in \Sigma(K)} \Lambda^+ .$$

The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\mathbb{X}^+ = \mathbb{X}(G)^+$ by its permutation of the summands; hence this action factors through an action of Γ. By [9], Thm. 7.2, the irreducible k-representations of G are indexed by the elements of $\mathbb{X}^+/\Gamma$. If α is a Γ-orbit in $\mathbb{X}^+$ we denote the corresponding irreducible representation of G by ρ_α.

We have a natural isomorphism $G_{\tilde{K}} \cong \prod_{\sigma \in \Sigma(K)} H_\sigma$, with $H_\sigma = H \otimes_{K,\sigma} \tilde{K}$. The representation $\rho_{\alpha,\tilde{K}}$ decomposes as a direct sum $\oplus_{\boldsymbol{\lambda} \in \alpha} \Psi_{\boldsymbol{\lambda}}$, where $\Psi_{\boldsymbol{\lambda}}$ is the external tensor product $\boxtimes_{\sigma \in \Sigma(K)} \psi_{\boldsymbol{\lambda}(\sigma)}$. (Here $\boldsymbol{\lambda} \in \mathbb{X}^+$ is viewed as a function $\Sigma(K) \to \Lambda^+$.)

Note that, since $G(k) = B^*$ is Zariski dense in G, the representations ρ_α, for $\alpha \in \mathbb{X}^+/\Gamma$, are still irreducible and mutually non-equivalent as representations of the abstract group B^*.

(1.5) Choose a k-basis $\{\beta_1, \dots, \beta_N\}$ for B (with $N = nd^2$). If E is a commutative k-algebra, we call a map $r\colon B \to E$ a multiplicative homogeneous polynomial map over k of degree i if it has the following properties:

(a) r is multiplicative, in the sense that $r(1) = 1$ and $r(b_1 b_2) = r(b_1) r(b_2)$ for all b_1, $b_2 \in B$;

(b) there exists a homogeneous polynomial $P \in E[t_1, \dots, t_N]$ of degree i such that $r(c_1\beta_1 + \cdots + c_N\beta_N) = P(c_1, \dots, c_N)$ for all $c_1, \dots, c_N \in k$.

Note that the polynomial P in (b) is uniquely determined, because k is an infinite field.

Let V be a finite dimensional k-vector space. Consider a multiplicative homogeneous polynomial map $r\colon B \to \mathrm{End}_k(V)$ over k of degree i. If R is a commutative k-algebra, define $r_R\colon B \otimes_k R \to \mathrm{End}_R(V \otimes_k R) = \mathrm{End}_k(V) \otimes_k R$ by the relation $r_R(c_1\beta_1 + \cdots + c_N\beta_N) = P(c_1, \dots, c_N)$, for $c_1, \dots, c_N \in R$. Using that r is multiplicative plus the fact that the field k is infinite, one easily shows that the map r_R is again multiplicative. Hence this construction defines an algebraic representation $\phi_r\colon G \to \mathrm{GL}(V)$ over k. We refer to the representations of G, or of $B^* = G(k)$, that are obtained in this manner as the polynomial representations of degree i.

(1.6) Define a subset $\Lambda^{\mathrm{pol}} \subset \Lambda^+$ by the condition that $\lambda_d \geqslant 0$, i.e.,

(1.6.1)
$$\Lambda^{\mathrm{pol}} = \left\{ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d \;\middle|\; \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d \geqslant 0 \right\}.$$

Define $\mathbb{X}^{\mathrm{pol}} = \oplus_{\sigma \in \Sigma(K)} \Lambda^{\mathrm{pol}}$, which is a Γ-stable subset of $\mathbb{X}^+$.

For $\boldsymbol{\lambda} \in \mathbb{X}^{\mathrm{pol}}$, define $\|\boldsymbol{\lambda}\| = \sum_{\sigma \in \Sigma(K)} \left|\boldsymbol{\lambda}(\sigma)\right|$. As the map $\mathbb{X}^{\mathrm{pol}} \to \mathbb{Z}_{\geqslant 0}$ given by $\boldsymbol{\lambda} \mapsto \|\boldsymbol{\lambda}\|$ is Γ-invariant, it descends to a map $\|\ \|\colon \mathbb{X}^{\mathrm{pol}}/\Gamma \to \mathbb{Z}_{\geqslant 0}$.

(1.7) *Proposition. — Let $\phi\colon B^* \to \mathrm{GL}(V)$ be a polynomial representation of degree i. Then*

(1.7.1)
$$(V, \phi) = \bigoplus_{\substack{\alpha \in \mathbb{X}^{\mathrm{pol}}/\Gamma \\ \|\alpha\| = i}} (V_\alpha, \phi^{(\alpha)})$$

such that $(V_\alpha, \phi^{(\alpha)})$ is isomorphic to a sum of copies of the irreducible representation ρ_α.

Proof. By construction, $\phi\colon B^* \to \mathrm{GL}(V)$ is obtained from an algebraic representation $\phi_r\colon G \to \mathrm{GL}(V)$ by evaluation on k-rational points. The irreducible representations that occur in ϕ_r are again polynomial of degree i, and this property is preserved if we extend scalars to $\bar{K}$. Using the description of the representations $\rho_{\alpha,\bar{K}}$ given in (1.3) and (1.4) we see that the only irreducible representations ρ_α that are polynomial of degree i are those with $\alpha \in \mathbb{X}^{\mathrm{pol}}/\Gamma$ and $\|\alpha\| = i$. $\square$

(1.8) *Example.* — The reduced norm $\mathrm{Nrd}\colon B^* \to k^*$ is a polynomial representation of degree nd. It corresponds to the Γ-orbit $\alpha \in \mathbb{X}^{\mathrm{pol}}/\Gamma$ that consists of the single element $\boldsymbol{\nu}\colon \Sigma(K) \to \Lambda^{\mathrm{pol}}$ with $\boldsymbol{\nu}(\sigma) = (1, \dots, 1)$ for all $\sigma \in \Sigma(K)$. If $\alpha \in \mathbb{X}^{\mathrm{pol}}/\Gamma$ is the orbit of $\boldsymbol{\lambda}\colon \Sigma(K) \to \Lambda^{\mathrm{pol}}$, the representation $\mathrm{Nrd} \otimes \rho_\alpha$ is again polynomial; it corresponds to the Γ-orbit in $\mathbb{X}^{\mathrm{pol}}$ of the sum $\boldsymbol{\nu} + \boldsymbol{\lambda}$.

(1.9) *Remark.* — We shall have to deal with multiplicative homogeneous polynomial maps $r\colon B \to \mathrm{End}_k(V)$ of degree i where V is no longer assumed to have finite k-dimension, but is the union of its finite dimensional subspaces V' that are stable under all operators $r(b)$ for $b \in B$. In this case we still have a decomposition (1.7.1), of course with the understanding that the

$(V_\alpha, \phi^{(\alpha)})$ will now in general be infinite sums of copies of ρ_α. We refer to V_α as the α-isotypic component of V.

2. Some preliminaries on the action of endomorphisms on the Chow motive

(2.1) Throughout this section, F is a field and S denotes a connected F-scheme that is smooth and quasi-projective over F. Let $\mathbf{M}^0(S)$ be the category of Chow motives over S with respect to graded correspondences, as defined as in [4], 1.6.

Let $\mathbf{V}_S$ denote the category of smooth projective S-schemes. We have a contravariant functor $R(-/S)\colon \mathbf{V}_S \to \mathbf{M}^0(S)$, sending a smooth projective $X \to S$ to $R(X/S) = (X, [{}^{\mathrm{t}}\Gamma_{\mathrm{id}}], 0)$. For a morphism $f\colon X \to Y$ between smooth projective S-schemes, $R(f/S) = [{}^{\mathrm{t}}\Gamma_f]\colon R(Y/S) \to R(X/S)$. We write f^* for $R(f/S)$.

Let $X \to S$ be an abelian scheme of relative dimension g over S. For $m \in \mathbb{Z}$, let $[m]_X\colon X \to X$ denote the multiplication by m map. By [4], Cor. 3.2 the relative motive $R(X/S)$ decomposes in $\mathbf{M}^0(S)$ as

$$\textbf{(2.1.1)} \qquad R(X/S) = \bigoplus_{i=0}^{2g} R^i(X/S)\,,$$

in such a way that $[m]_X^*$ acts on $R^i(X/S)$ as multiplication by m^i. Define $R^i(X/S) = 0$ if $i \notin \{0, \ldots, 2g\}$. If $f\colon X \to Y$ is a homomorphism of abelian schemes over S the induced morphism f^* of motives is a sum of morphisms $R^i(f)\colon R^i(Y/S) \to R^i(X/S)$; we shall again call these morphisms f^*.

The goal of this paper is to explain how, in the presence of non-trivial endomorphisms, the decomposition (2.1.1) may be refined. As a first example we consider the case of a product of abelian schemes. Though it is not stated by Deninger and Murre in [4], the following result is an immediate consequence of their work.

(2.2) *Proposition.* — *Let $X_1, \ldots, X_r$ be abelian schemes over S with X_ν of relative dimension g_ν. Write $X = X_1 \times_S \cdots \times_S X_r$, let $g = g_1 + \cdots + g_r$ and*

$$I_X = \bigl\{\boldsymbol{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r \bigm| 0 \leqslant i_\nu \leqslant 2g_\nu\bigr\}\,.$$

For $\boldsymbol{m} = (m_1, \ldots, m_r)$, let $[\boldsymbol{m}]_X \in \mathrm{End}(X/S)$ be given by $(x_1, \ldots, x_r) \mapsto (m_1x_1, \ldots, m_rx_r)$, and let $\boldsymbol{m}^{\boldsymbol{i}} = m_1^{i_1} \cdots m_r^{i_r}$. Then there is a unique decomposition

$$\textbf{(2.2.1)} \qquad [\Delta_{X/S}] = \sum_{\boldsymbol{i} \in I_X} \pi_{\boldsymbol{i}}$$

in $\mathrm{End}_{\mathbf{M}^0(S)}\bigl(R(X/S)\bigr) = \mathrm{CH}^g(X \times_S X)$ such that the elements $\pi_{\boldsymbol{i}}$ are mutually orthogonal idempotents and such that $[\boldsymbol{m}]_X^* \circ \pi_{\boldsymbol{i}} = \boldsymbol{m}^{\boldsymbol{i}} \cdot \pi_{\boldsymbol{i}}$ for all $\boldsymbol{m} \in \mathbb{Z}^r$ and $\boldsymbol{i} \in I_X$. Moreover, $\pi_{\boldsymbol{i}} \circ [\boldsymbol{m}]_X^* = \boldsymbol{m}^{\boldsymbol{i}} \cdot \pi_{\boldsymbol{i}}$ for all $\boldsymbol{m}$ and $\boldsymbol{i}$. Corresponding to (2.2.1) we have a decomposition

$$R(X/S) = \bigoplus_{\boldsymbol{i} \in I_X} R^{\boldsymbol{i}}(X/S)$$

such that $[\boldsymbol{m}]^*$ acts on $R^{\boldsymbol{i}}(X/S)$ as multiplication by $\boldsymbol{m}^{\boldsymbol{i}}$.

Proof. This follows from the main results of [4] by taking tensor products. We have $R(X/S) = R(X_1/S) \otimes \cdots \otimes R(X_r/S)$ in $\mathbf{M}^0(S)$; if $[\Delta_{X_\nu/S}] = \sum_{j=0}^{2g_\nu} \pi_j^{(\nu)}$ is the decomposition of loc. cit., Thm. 3.1., we take $\pi_{\boldsymbol{i}} = \pi_{i_1}^{(1)} \otimes \cdots \otimes \pi_{i_r}^{(r)}$ for $\boldsymbol{i} = (i_1, \ldots, i_r) \in I_X$. □

(2.3) *Example.* — (Cf. [7], (3.1.2)(ii).) Let X and Y be abelian schemes over S with X of relative dimension g. If $\xi \in \mathrm{CH}(X \times_S Y)$ we have a decomposition $\xi = \sum \xi_{i,j}$ such that $[m,n]^*(\xi_{i,j}) = m^i n^j \cdot \xi_{i,j}$ for all integers m and n. It follows from the relations in [4], Prop. 1.2.1, together with the motivic Poincaré duality ${}^{\mathrm{t}}\pi_i = \pi_{2g-i}$ that $\xi_{i,j} = \pi_j(Y/S) \circ \xi \circ \pi_{2g-i}(X/S)$.

We apply this with $Y = X^\dagger$, the dual of X. Let $\ell \in \mathrm{CH}^1(X \times_S X^\dagger)$ be the first Chern class of the Poincaré bundle. Then $\ell = \ell_{1,1}$; hence, $\ell^i/i! = \pi_i(X^\dagger/S) \circ \big(\ell^i/i!\big) \circ \pi_{2g-i}(X/S)$. Now use the Mukai-Beauville relation $\mathscr{F}^\dagger \circ \mathscr{F} = (-1)^g[-1]^*$ and view $\ell^i/i! \in \mathrm{CH}^i(X \times_S X^\dagger)$ as a morphism from $R(X/S) = \oplus R^j(X/S)$ to $R(X^\dagger/S)\big(g-i\big) = \oplus R^j(X^\dagger/S)\big(i-g\big)$. It follows that the only non-zero component of this morphism is an isomorphism

$$\textbf{(2.3.1)} \qquad \frac{\ell^i}{i!}\colon R^{2g-i}(X/S) \xrightarrow{\sim} R^i(X^\dagger/S)\big(i-g\big)\,,$$

which we refer to as motivic Fourier duality. (The interpretation is that, up to a Tate twist, the dual abelian scheme is the Poincaré dual of X, and that Fourier duality "is" Poincaré duality. Indeed, combining (2.3.1) with the motivic Poincaré duality $R^i(X/S)^\vee = R^{2g-i}(X/S)\big(g\big)$ we find that $R^i(X^\dagger/S) \cong R^i(X/S)^\vee\big(-i\big)$.)

(2.4) With S as in (2.1), consider an abelian scheme $X \to S$ of relative dimension $g > 0$. We assume X is isogenous to a power of a simple abelian scheme over S, in which case the endomorphism algebra $D = \mathrm{End}(X/S) \otimes \mathbb{Q}$ is a simple $\mathbb{Q}$-algebra of finite dimension. (For the general case see (5.5).) Let Let K be the center of D. Let $n = [K : \mathbb{Q}]$ and $d = \dim_K(D)^{1/2}$. Let $\Sigma(K)$ be the set of ring homomorphisms $K \to \overline{\mathbb{Q}}$, let $\tilde{K} \subset \overline{\mathbb{Q}}$ denote the normal closure of K inside $\overline{\mathbb{Q}}$, and write $\Gamma = \mathrm{Gal}(\tilde{K}/\mathbb{Q})$.

Every element of D can be written in the form f/m for some $f \in \mathrm{End}(X/S)$ and some integer $m \neq 0$. For $i \geqslant 0$ we have a well-defined map $r^{(i)}\colon D^{\mathrm{opp}} \to \mathrm{End}_{\mathbf{M}^0(S)}\big(R^i(X/S)\big)$ given by $(f/m) \mapsto f^* \circ [m]^{*,-1}$. This map is multiplicative but is not, in general, additive. In particular, the group $D^{\mathrm{opp},*}$ acts on $R^i(X/S)$ by automorphisms.

(2.5) *Proposition.* — *The map $r^{(1)}\colon D^{\mathrm{opp}} \to \mathrm{End}_{\mathbf{M}^0(S)}\big(R^1(X/S)\big)$ is a homomorphism of $\mathbb{Q}$-algebras.*

Proof. It will be easier to prove the dual statement. Recall that $R^{2g-1}(X/S)\big(g\big) = R^1(X/S)^\vee$; see [7], (3.1.2). If f is an endomorphism of X/S, we have an induced endomorphism $f_* = \big[\Gamma_f\big]$ of $R(X/S)$. It follows from Prop. 3.3 of [4], taking transposes, that $\pi_i \circ f_* = f_* \circ \pi_i$ for all i. Hence f_* is the sum of the endomorphisms $f_* \circ \pi_i \in \mathrm{End}_{\mathbf{M}^0(S)}\big(R^i(X/S)\big)$; we shall again denote these by f_*. For $m \in \mathbb{Z}$ the endomorphism $[m]_*\colon R^i(X/S) \to R^i(X/S)$ is the multiplication by m^{2g-i}; hence for $m \neq 0$ it is an isomorphism and we can define a map $D \to \mathrm{End}_{\mathbf{M}^0(S)}\big(R^{2g-1}(X/S)\big)$ by $(f/m) \mapsto f_* \circ [m]_*^{-1}$. It suffices to prove that this map is additive.

Let $A \to T$ be an abelian scheme of relative dimension g with T a connected, smooth and quasi-projective F-scheme. For $a \in A(T)$, define $\log\big([\Gamma_a]\big) \in \mathrm{CH}^g(A)$ as in [7], Section (1.4).

As shown there, $\log\bigl([\Gamma_{a+b}]\bigr) = \log\bigl([\Gamma_a]\bigr) + \log\bigl([\Gamma_b]\bigr)$. Applying this to the abelian scheme $\mathrm{pr}_1\colon X \times_S X \to X$ we find that for endomorphisms f and f' of X/S we have

$$\textbf{(2.5.1)} \qquad \log\bigl([\Gamma_{f+f'}]\bigr) = \log\bigl([\Gamma_f]\bigr) + \log\bigl([\Gamma_{f'}]\bigr)$$

in $\mathrm{CH}^g(X \times_S X) = \mathrm{End}_{\mathbf{M}^0(S)}\bigl(R(X/S)\bigr)$.

The projector π_{2g-1} that defines $R^{2g-1}(X/S)$ is $\pi_{2g-1} = \log\bigl([\Gamma_{\mathrm{id}}]\bigr)$. Now we use [6], assertion (iii) of Lemma 2.2; this says that for an endomorphism ϕ we have $\phi_* \circ \log\bigl([\Gamma_{\mathrm{id}}]\bigr) = \log\bigl([\Gamma_\phi]\bigr)$. So (2.5.1) gives $(f+f')_* \circ \pi_{2g-1} = f_* \circ \pi_{2g-1} + f'_* \circ \pi_{2g-1}$, which is what we wanted to prove. □

(2.6) *Corollary. — The map* $r^{(i)}\colon D^{\mathrm{opp}} \to \mathrm{End}_{\mathbf{M}^0(S)}\bigl(R^i(X/S)\bigr)$ *defined in* (2.4) *is a multiplicative homogeneous polynomial map over* $\mathbb{Q}$ *of degree* i.

Proof. We already know that $r^{(i)}$ is multiplicative. Taking the isomorphism $R^i(X/S) \xrightarrow{\sim} \wedge^i R^1(X/S)$ of [7], Thm. (3.3.1), as an identification, the map $r^{(i)}$ is the composition of the homomorphism $r^{(1)}$ with the map $\mathrm{End}_{\mathbf{M}^0(S)}\bigl(R^1(X/S)\bigr) \to \mathrm{End}_{\mathbf{M}^0(S)}\bigl(R^i(X/S)\bigr)$ that sends an endomorphism h of $R^1(X/S)$ to the induced endomorphism $\wedge^i h = h \wedge \cdots \wedge h$ of $R^i(X/S)$. It follows that $r^{(i)}$ is a homogeneous polynomial map of degree i. □

3. Duality

(3.1) We retain the notation and assumptions of (2.4). We apply the theory of Section 1 with $k = \mathbb{Q}$ and three different choices for B, to be discussed in more detail below. In each case B is central simple of dimension d^2 over the field K of (2.4). The meaning of $\Sigma(K)$ and Γ is the same in all cases and the notation we use is consistent with the notation introduced in Section 1. In each case we index the irreducible algebraic representations of B^* by $\mathbb{X}^+/\Gamma$, following the method discussed in (1.2)–(1.4).

Let us now give some more details about the group actions we consider.

(a) We shall mostly describe things from the cohomological perspective. In this case we take $B = D^{\mathrm{opp}}$, which we let act on $\mathrm{CH}(X)$ through the operators f^*. Let H denote the reductive group over K with $H(R) = (D^{\mathrm{opp}} \otimes_K R)^*$ and let $G = \mathrm{Res}_{K/\mathbb{Q}} H$. For $\lambda \in \Lambda^+$, let ψ_λ be the corresponding irreducible representation of H over K. For $\alpha \in \mathbb{X}^+/\Gamma$, let ρ_α be the corresponding irreducible representation of $G(\mathbb{Q}) = D^{\mathrm{opp},*}$ over $\mathbb{Q}$.

(b) In order to describe Poincaré duality we need the homological perspective, letting $B = D$ act on $\mathrm{CH}(X)$ through the operators f_*. Let H' be the reductive group over K with $H'(R) = (D \otimes_K R)^*$ and let $G' = \mathrm{Res}_{K/\mathbb{Q}} H'$, which is the opposite of the group G. For $\lambda \in \Lambda^+$, let ψ'_λ be the corresponding irreducible representation of H' over K. For $\alpha \in \mathbb{X}^+/\Gamma$, the corresponding irreducible representation of $G'(\mathbb{Q}) = D^*$ over $\mathbb{Q}$ is denoted by ρ'_α.

(c) Let $X^\dagger \to S$ be the dual abelian scheme and let $D^\dagger = \mathrm{End}(X^\dagger/S) \otimes \mathbb{Q}$. If f is an endomorphism of X/S, let $f^\dagger\colon X^\dagger \to X^\dagger$ denote the dual endomorphism. The map $f \mapsto f^\dagger$ gives an isomorphism of $\mathbb{Q}$-algebras $D \xrightarrow{\sim} D^{\dagger,\mathrm{opp}}$ and we use this to identify the center of $D^{\dagger,\mathrm{opp}}$ with K. (This may lead to confusion; see (3.5).) For the rest the pattern is the same as in (a). We consider $\mathrm{CH}(X^\dagger)$ as a representation of $D^{\dagger,\mathrm{opp},*}$, with $g \in D^{\dagger,\mathrm{opp}}$ acting as g^*. For $\alpha \in \mathbb{X}^+/\Gamma$, let $\rho^\dagger_\alpha$ be the corresponding irreducible representation of $D^{\dagger,\mathrm{opp},*}$ over $\mathbb{Q}$.

(3.2) *Lemma.* — *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ be an element of Λ^+. Then the representation τ of H' over K given by $\tau(h) = \psi_\lambda(h^{-1})$ is isomorphic to ψ'_μ, where $\mu = (-\lambda_d, \ldots, -\lambda_1)$.*

Proof. It is clear that τ is an irreducible representation of H'. As the representations are determined by their highest weights, we may work over $\bar{K}$. Choose an isomorphism of $\bar{K}$-algebras $a\colon D^{\mathrm{opp}}_{\bar{K}} \xrightarrow{\sim} M_d(\bar{K})$, and define $a'\colon D_{\bar{K}} \xrightarrow{\sim} M_d(\bar{K})$ by $a'(\xi) = {}^{\mathrm{t}}a(\xi)$, the transpose of $a(\xi)$. Let $\alpha\colon H_{\bar{K}} \xrightarrow{\sim} \mathrm{GL}_{d,\bar{K}}$ and $\alpha'\colon H'_{\bar{K}} \xrightarrow{\sim} \mathrm{GL}_{d,\bar{K}}$ be the induced isomorphisms of algebraic groups. Via these isomorphisms we can view both ψ_λ and τ as representations of $\mathrm{GL}_{d,\bar{K}}$; in other words, we consider $\psi_\lambda \circ \alpha^{-1}$ and $\tau \circ (\alpha')^{-1}$. In both cases the highest weight is taken with regard to the diagonal torus T and the upper triangular Borel $Q \subset \mathrm{GL}_d$. We have

$$\bigl(\tau \circ (\alpha')^{-1}\bigr)(g) = \bigl(\psi_\lambda \circ \alpha^{-1}\bigr)({}^{\mathrm{t}}g^{-1}) .$$

Let β be the automorphism of GL_d given by $g \mapsto {}^{\mathrm{t}}g^{-1}$. Then $\beta(T) = T$ and $\beta(Q) = Q^-$, the lower triangular Borel subgroup. If $A \in \mathrm{GL}_d(K)$ is the anti-diagonal matrix with all anti-diagonal coefficients equal to 1, the inner automorphism $\mathrm{Inn}(A)$ transforms (T, Q^-) back to (T, Q), and the effect of $\mathrm{Inn}(A) \circ \beta$ on the character group of T is given by $e_i \mapsto -e_{d-i}$. Hence if $\psi_\lambda \circ \alpha^{-1}$ has highest weight $\lambda_1 e_1 + \cdots + \lambda_d e_d$, the highest weight of $\tau \circ (\alpha')^{-1}$ is $-\lambda_d e_1 - \cdots - \lambda_1 e_d$. $\square$

(3.3) *Notation.* — For $\lambda = (\lambda_1, \ldots, \lambda_d)$ in Λ^+ define

$$\lambda^\star = \left(\frac{2g}{nd} - \lambda_d, \ldots, \frac{2g}{nd} - \lambda_1\right) .$$

Note that $2g/nd$ is an integer; see [8], Chap. 19, Corollary to Thm. 4. Hence $\lambda^\star$ is again an element of Λ^+. For $\boldsymbol{\lambda} \in \mathbb{X}^+$, define $\boldsymbol{\lambda}^\star \in \mathbb{X}^+$ by the rule $\boldsymbol{\lambda}^\star(\sigma) = \boldsymbol{\lambda}(\sigma)^\star$. For $\alpha \in \mathbb{X}^+/\Gamma$, let $\alpha^\star$ denote the Γ-orbit consisting of the elements $\boldsymbol{\lambda}^\star$, for $\boldsymbol{\lambda} \in \alpha$. Note that $\|\alpha^\star\| = 2g - \|\alpha\|$.

(3.4) *Proposition.* — *Let $V \subset \mathrm{CH}(X)$ be an irreducible subrepresentation of $D^{\mathrm{opp},*}$ that is isomorphic to ρ_α.*

(i) *The subspace $V \subset \mathrm{CH}(X)$ is stable under the action of the operators f_*, for $f \in D$, and V is isomorphic to $\rho'_{\alpha^\star}$ as a representation of D^*.*

(ii) *Let $\mathscr{F}\colon \mathrm{CH}(X) \xrightarrow{\sim} \mathrm{CH}(X^\dagger)$ be the Fourier transform. Then $\mathscr{F}(V) \subset \mathrm{CH}(X^\dagger)$ is an irreducible subrepresentation of $D^{\dagger,\mathrm{opp},*}$ that is isomorphic to $\rho^\dagger_{\alpha^\star}$.*

Proof. (i) Let $f \in D^{\mathrm{opp},*}$. Then f is a quasi-isogeny of X to itself. Its degree $\deg(f)$ equals $\mathrm{Nrd}(f)^{(2g/nd)}$, where Nrd: $D^{\mathrm{opp},*} \to \mathbb{Q}^*$ is the reduced norm character. (See (1.8).) For $\xi \in \mathrm{CH}(X)$ we have the relation $f_*(\xi) = \deg(f) \cdot (1/f)^*\bigl(\xi\bigr)$. Now use (1.8) and Lemma (3.2).

(ii) For $f \in D$ and $\xi \in \mathrm{CH}(X)$ we have the relation $\mathscr{F}\bigl(f_*(\xi)\bigr) = f^{\dagger,*}\bigl(\mathscr{F}(\xi)\bigr)$. So (ii) follows from (i). $\square$

(3.5) *Caution.* — The field K is either totally real or a CM field. In (ii) of the Proposition, it is important that we identify K with the center of $D^{\dagger,\mathrm{opp}}$ via the isomorphism $D \xrightarrow{\sim} D^{\dagger,\mathrm{opp}}$ given by $f \mapsto f^\dagger$. If we choose a polarization $\theta\colon X \to X^\dagger$, the resulting isomorphism $D \xrightarrow{\sim} D^\dagger$ gives the complex conjugate identification of K with the center of $D^{\dagger,\mathrm{opp}}$. Under that identification, the Fourier dual of a $D^{\mathrm{opp},*}$-subrepresentation $V \subset \mathrm{CH}(X)$ of type ρ_α is a $D^{\dagger,\mathrm{opp},*}$-subrepresentation $\mathscr{F}(V) \subset \mathrm{CH}(X^\dagger)$ of type $\rho^\dagger_{\bar{\alpha}^\star}$, where $\bar{\alpha}^\star \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ is the complex conjugate of $\alpha^\star$.

4. Decomposition of the Chow ring

Notation and assumptions as in (2.4) and (3.1).

(4.1) *Lemma.* — *Let $U \subset \mathrm{CH}(X)$ be a $\mathbb{Q}$-subspace of finite dimension. Then the $\mathbb{Q}$-linear span of the classes $f^*(u)$, for $f \in D$ and $u \in U$, again has finite $\mathbb{Q}$-dimension.*

Proof. It suffices to prove this if $U = \mathbb{Q} \cdot u$ for some element $u \in \mathrm{CH}(X)$. Using the Deninger-Murre decomposition (2.1.1) we may, in addition, assume there is an integer i such that $[m]^*(u) = m^i \cdot u$ for all $m \in \mathbb{Z}$.

Choose a $\mathbb{Q}$-basis $\{\beta_1, \ldots, \beta_N\}$ of D with $\beta_1 = \mathrm{id}_X$. With $\mu\colon X^N \to X$ the addition map, consider the $\mathbb{Q}$-subspace of $\mathrm{CH}(X^N)$ spanned by the class $(\beta_1 \times \cdots \times \beta_N)^* \mu^*(u)$. By Prop. (2.2), applied to X^N, there exists a finite dimensional $\mathbb{Q}$-subspace $W \subset \mathrm{CH}(X^N)$ that contains all classes $(m_1\beta_1 \times \cdots \times m_N\beta_N)^*\mu^*(u)$ for $(m_1, \ldots, m_N) \in \mathbb{Z}^N$. Our assumptions on u imply that W even contains all $(q_1\beta_1 \times \cdots \times q_N\beta_N)^*\mu^*(u)$ for $(q_1, \ldots, q_N) \in \mathbb{Q}^N$. If $\Delta\colon X \to X^N$ is the diagonal morphism, $\Delta^*(W)$ is then a finite dimensional subspace of $\mathrm{CH}(X)$ that contains all classes $(q_1\beta_1 + \cdots + q_N\beta_N)^*(u)$, and because $\beta_1 = \mathrm{id}_X$ we have $U \subset \Delta^*(W)$. $\square$

(4.2) Define a subset $\Lambda^{\mathrm{adm}} \subset \Lambda^{\mathrm{pol}}$ of "admissible" elements by the condition that $(2g/nd) \geqslant \lambda_1$; so,

$$\Lambda^{\mathrm{adm}} = \Bigl\{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d \Bigm| \frac{2g}{nd} \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d \geqslant 0\Bigr\}.$$

Define $\mathbb{X}^{\mathrm{adm}} = \oplus_{\sigma\in\Sigma(K)} \Lambda^{\mathrm{adm}}$, which is a Γ-stable subset of $\mathbb{X}^+$. Note that $0 \leqslant \|\alpha\| \leqslant 2g$ for all $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$. If $\boldsymbol{\lambda} \in \mathbb{X}^{\mathrm{adm}}$ then $\boldsymbol{\lambda}^\star$ is an element of $\mathbb{X}^{\mathrm{adm}}$, too; hence $\alpha \mapsto \alpha^\star$ is an involutive automorphism of $\mathbb{X}^{\mathrm{adm}}/\Gamma$.

(4.3) *Theorem.* — *We have a decomposition*

$$\mathrm{CH}(X) = \bigoplus_{\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma} \mathrm{CH}_\alpha(X) \tag{4.3.1}$$

as a representation of $D^{\mathrm{opp},*}$, such that the $\mathrm{CH}_\alpha(X)$ is isomorphic to a sum of copies of the irreducible representation ρ_α. For $i \geqslant 0$ the subspace $\mathrm{CH}\bigl(R^i(X/S)\bigr) \subset \mathrm{CH}(X)$ is the direct sum of the $\mathrm{CH}_\alpha(X)$ with $\|\alpha\| = i$. For $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$, the Fourier transform $\mathscr{F}$ restricts to an isomorphism

$$\mathscr{F}\colon \mathrm{CH}_\alpha(X) \xrightarrow{\sim} \mathrm{CH}_{\alpha^\star}(X^\dagger).$$

Proof. By (2.6) and (4.1) we can apply Prop. (1.7). This gives a decomposition of $\mathrm{CH}\bigl(R^i(X/S)\bigr)$ as a direct sum of subspaces $\mathrm{CH}_\alpha\bigl(R^i(X/S)\bigr)$ for $\alpha \in \mathbb{X}^{\mathrm{pol}}/\Gamma$ with $\|\alpha\| = i$. (Cf. (1.9).) But if $\mathrm{CH}_\alpha\bigl(R^i(X/S)\bigr) \neq 0$ then it follows from Prop. (3.4) that $\alpha^\star$ lies in the subset $\mathbb{X}^{\mathrm{pol}}/\Gamma \subset \mathbb{X}^+/\Gamma$. This implies that $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$. The last assertion is immediate from (3.4). $\square$

5. Motivic decomposition

We retain the notation and assumptions of the previous sections; in particular, X/S is still assumed to be isogenous to a power of a simple abelian scheme.

(5.1) *Theorem.* — (i) *There is a unique decomposition*

$$R(X/S) = \bigoplus_{\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma} R^{(\alpha)}(X/S)\,, \tag{5.1.1}$$

in $\mathbf{M}^0(S)$ *that is stable under the action of* $D^{\mathrm{opp},*}$ *and has the property that for any* M *in* $\mathbf{M}^0(S)$ *the* $D^{\mathrm{opp},*}$*-representation* $\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M, R^{(\alpha)}(X/S)\bigr)$ *is a sum of copies of the irreducible representation* ρ_α. *The motive* $R^i(X/S)$ *is the direct sum of the* $R^{(\alpha)}(X/S)$ *with* $\|\alpha\| = i$.

(ii) *For* $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ *the subspace* $\mathrm{CH}\bigl(R^{(\alpha)}(X/S)\bigr) \subset \mathrm{CH}(X)$ *is the* α*-isotypic component* $\mathrm{CH}_\alpha(X) \subset \mathrm{CH}(X)$ *of* (4.3.1).

(iii) *Let* δ_α *be the idempotent in* $\mathrm{CH}^g(X \times_S X) = \mathrm{End}_{\mathbf{M}^0(S)}\bigl(R(X/S)\bigr)$ *that defines the submotive* $R^{(\alpha)}(X/S)$, *so that* $[\Delta_{X/S}] = \sum_{\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma} \delta_\alpha$ *is the decomposition of the diagonal that corresponds with* (5.1.1). *Then* ${}^{\mathrm{t}}\delta_\alpha = \delta_{\alpha^\star}$*; hence*

$$R^{(\alpha)}(X/S)^\vee = R^{(\alpha^\star)}(X/S)\bigl(g\bigr)\,.$$

(iv) *The motivic Fourier duality* $R^{2g-i}(X/S) \xrightarrow{\sim} R^i(X^\dagger/S)\bigl(i-g\bigr)$ *of* (2.3.1) *is the direct sum of isomorphisms*

$$R^{(\alpha)}(X/S) \xrightarrow{\sim} R^{(\alpha^\star)}(X^\dagger/S)\bigl(i-g\bigr)$$

for $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ *with* $\|\alpha\| = 2g-i$.

Proof. (i) We view $X \times_S X$ as an abelian scheme over X via the first projection. Correspondingly, we let an element $f \in D$ act on $\mathrm{CH}(X \times_S X)$ as $(1 \times f)^*$. By Thm. (4.3),

$$\mathrm{CH}(X \times_S X) = \bigoplus_{\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma} \mathrm{CH}_\alpha(X \times_S X) \tag{5.1.2}$$

such that $\mathrm{CH}_\alpha(X \times_S X)$ is α-isotypic as a representation of $D^{\mathrm{opp},*}$. For m an integer, $\bigl(1 \times [m]\bigr)^*$ is multiplication by $m^{\|\alpha\|}$ on $\mathrm{CH}_\alpha(X \times_S X)$; hence the idempotent π_i lies in the direct sum of the subspaces $\mathrm{CH}_\alpha(X \times_S X)$ with $\|\alpha\| = i$. Define δ_α to be the α-component of $[\Delta_{X/S}]$ in (5.1.2).

For $\xi \in \mathrm{CH}(X \times_S X)$ let $W(\xi) \subset \mathrm{CH}(X \times_S X)$ denote the smallest $\mathbb{Q}$-subspace containing ξ that is stable under the action of $D^{\mathrm{opp},*}$, i.e., the linear span of the elements $(1 \times f)^*\xi$, for $f \in D^{\mathrm{opp},*}$. If ξ and η are correspondences from X to itself relative to S and $f \in D$, it follows from [4], Prop. 1.2.1, that $(1 \times f)^*\bigl(\eta \circ \xi\bigr) = (1 \times f)^*\eta \circ \xi$. Hence $W(\eta \circ \xi)$, as a representation of $D^{\mathrm{opp},*}$, is a quotient of $W(\eta)$. In particular, for $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ we have $\delta_\alpha \circ \xi \in \mathrm{CH}_\alpha(X \times_S X)$. On the other hand, $\xi = [\Delta_{X/S}] \circ \xi = \sum_{\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma} \delta_\alpha \circ \xi$; hence $\delta_\alpha \circ \xi$ is the α-component of ξ in the decomposition (5.1.2). It follows that

$$\delta_\beta \circ \delta_\alpha = \begin{cases} \delta_\alpha & \text{if } \beta = \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, δ_α is an idempotent. Define $R^{(\alpha)}(X/S) = (X, \delta_\alpha, 0)$, the submotive of $R(X/S)$ cut out by δ_α. By construction we have a decomposition (5.1.1). Further, $(1 \times f)^*$ preserves the

subspaces $\mathrm{CH}_\beta(X\times_S X)\subset \mathrm{CH}(X\times_S X)$; so $(1\times f)^*\bigl(\delta_\beta\bigr) = \bigl[{}^{\mathrm{t}}\Gamma_f\bigr]\circ\delta_\beta$ lies in $\mathrm{CH}_\beta(X\times_S X)$, and by the above it follows that $\delta_\alpha\circ\bigl[{}^{\mathrm{t}}\Gamma_f\bigr]\circ\delta_\beta = 0$ if $\alpha\neq\beta$. This means that the decomposition (5.1.1) is stable under the action of $D^{\mathrm{opp},*}$.

If M is a relative Chow motive over S the map $h\mapsto \sum\delta_\alpha\circ h$ gives an isomorphism $\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M,R(X/S)\bigr)\xrightarrow{\sim}\oplus_{\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma}\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M,R^{(\alpha)}(X/S)\bigr)$. By the same argument as above, the $D^{\mathrm{opp},*}$-subrepresentation of $\mathrm{Hom}_{\mathbf{M}^0(S)}\bigl(M,R^{(\alpha)}(X/S)\bigr)$ generated by $\delta_\alpha\circ h$ is α-isotypic.

Finally, the uniqueness of the decomposition (5.1.1) follows from the Yoneda Lemma, as the required property uniquely characterizes $R^{(\alpha)}(X/S)$ as a subfunctor of $R(X/S)$.

Part (ii) follows from (i) by taking $M=\mathbf{1}(-j)$ for various j.

Next we prove (iv). Given a motive M and a class $\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma$ with $\|\alpha\| = 2g-i$, consider the map $h\colon \mathrm{Hom}\bigl(M,R^{(\alpha)}(X/S)\bigr)\to\mathrm{Hom}\bigl(M,R(X^\dagger/S)(i-g)\bigr)$ induced by the composition

$$R^{(\alpha)}(X/S)\hookrightarrow R^{2g-i}(X/S)\xrightarrow{(2.3.1)} R^i(X^\dagger/S)\bigl(i-g\bigr)\hookrightarrow R(X^\dagger/S)\bigl(i-g\bigr)\,.$$

By Yoneda, it suffices to prove that the image of h lies in the $\alpha^\star$-isotypic component of $\mathrm{Hom}\bigl(M,R(X^\dagger/S)(i-g)\bigr)$. It is enough to do this for motives M of the form $M=R(Y/S)\bigl(m\bigr)$ with Y a connected smooth projective S-scheme. In this case, h is just the Fourier transform $\mathrm{CH}_\alpha^{\dim(Y/S)-m}(Y\times_S X)\to\mathrm{CH}^{\dim(Y/S)-m-g+i}(Y\times_S X^\dagger)$, where we view $Y\times_S X$ and $Y\times_S X^\dagger$ as abelian schemes over Y via the first projections. (We use that our motivic decomposition is compatible, in the obvious sense, with base-change.) We conclude by Thm. (4.3).

For (iii) we first recall from (3.1)(c) that we have a natural isomorphism $\tau\colon D^*\cong D^{\dagger,\mathrm{opp},*}$. On $R^i(X/S)^\vee$ we have an action of D^*. On $R^i(X^\dagger/S)\bigl(i\bigr)$ we have an action of $D^{\dagger,\mathrm{opp},*}$. Further, the isomorphism $R^i(X/S)^\vee\xrightarrow{\sim}R^i(X^\dagger/S)\bigl(i\bigr)$ of (2.3)) is equivariant with respect to τ. (Cf. the proof of (3.4)(ii).) With these remarks, (iii) follows from (iv). $\square$

(5.2) *Corollary. — Let* $\mathbf{Vect}_\mathbb{Q}$ *be the category of* $\mathbb{Q}$*-vector spaces. If* $\Phi\colon \mathbf{M}^0(S)\to\mathbf{Vect}_\mathbb{Q}$ *is a* $\mathbb{Q}$*-linear functor,* $\Phi\bigl(R(X/S)\bigr) = \oplus_{\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma}\Phi\bigl(R^{(\alpha)}(X/S)\bigr)$ *and* $\Phi\bigl(R^{(\alpha)}(X/S)\bigr)$ *is* α*-isotypic as a representation of* $D^{\mathrm{opp},*}$*.*

Proof. Write $R^{(\alpha)}$ for $R^{(\alpha)}(X/S)$. Let $E_\alpha\subset\mathrm{End}_{\mathbf{M}^0(S)}(R^{(\alpha)})$ be the image of the group algebra $\mathbb{Q}[D^{\mathrm{opp},*}]$, or, what is the same, the $D^{\mathrm{opp},*}$-subrepresentation of $\mathrm{End}_{\mathbf{M}^0(S)}(R^{(\alpha)})$ generated by the identity. If $u\in\Phi(R^{(\alpha)})$, the $D^{\mathrm{opp},*}$-subrepresentation of $\Phi(R^{(\alpha)})$ generated by u is a quotient of E_α. Now use that $\mathrm{End}_{\mathbf{M}^0(S)}(R^{(\alpha)})$ is α-isotypic as a representation of $D^{\mathrm{opp},*}$. $\square$

(5.3) *Example.* — For the higher Chow groups (with $\mathbb{Q}$-coefficients) we have

$$\mathrm{CH}(X;j) = \bigoplus_{\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma}\mathrm{CH}\bigl(R^{(\alpha)}(X/S);j\bigr)$$

and $\mathrm{CH}\bigl(R^{(\alpha)}(X/S);j\bigr)$ is α-isotypic as a representation of $D^{\mathrm{opp},*}$.

Depending on the context we can draw similar conclusions for cohomology. For instance, if the ground field F is $\mathbb{C}$ and if $q\colon X\to S$ is the structural morphism, the variation of Hodge structure $\mathbb{V} = R^nq_*\mathbb{Q}_X$ decomposes as a direct sum $\oplus_{\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma}\mathbb{V}_\alpha$ where $\mathbb{V}_\alpha\subset\mathbb{V}$ is cut out by the projector δ_α and is α-isotypic as a sheaf of $D^{\mathrm{opp},*}$-modules.

If we have a cohomology theory with coefficients in a field $\mathbb{F}$ of characteristic 0, we can in general only conclude that the cohomology of $R^{(\alpha)}(X/S)$ is a quotient of a sum of copies of $\rho_{\alpha,\mathbb{F}}$. For instance, if E is a supersingular elliptic curve over $\overline{\mathbb{F}}_p$, in which case D is a quaternion algebra over $\mathbb{Q}$, there is a unique class $\alpha \in \mathbb{X}^{\mathrm{adm}}/\Gamma$ with $\|\alpha\| = 1$ (see also below) and ρ_α has dimension 4; so the ℓ-adic cohomology $H^1(E, \mathbb{Q}_\ell)$ is only "half" a copy of $\rho_{\alpha,\mathbb{Q}_\ell}$.

(5.4) *Example.* — Suppose D is a quaternion algebra with center $\mathbb{Q}$. In this case $\mathbb{X}(G)^{\mathrm{adm}}/\Gamma$ is the set of pairs $\lambda = (\lambda_1, \lambda_2)$ with $g \geqslant \lambda_1 \geqslant \lambda_2 \geqslant 0$. Viewing $D^{\mathrm{opp},*}$ as an inner form of GL_2 over $\mathbb{Q}$, the irreducible representation ρ_λ associated with λ (which in this case is the same as the representation ψ_λ of (1.3)) is a $\mathbb{Q}$-form of $d(\lambda)$ copies of the representation $\mathrm{Sym}^{\lambda_1-\lambda_2}(V) \otimes \det^{\otimes \lambda_2}$, where V is the standard representation of GL_2 and where

$$d(\lambda) = \begin{cases} 1 & \text{if } \lambda_1 - \lambda_2 \text{ is even;} \\ 2 & \text{if } \lambda_1 - \lambda_2 \text{ is odd.} \end{cases}$$

For $0 \leqslant i \leqslant g$ we obtain a decomposition

$$R^i(X/S) = R^{(i,0)} \oplus R^{(i-1,1)} \oplus \cdots \oplus R^{(\nu,i-\nu)} \quad \text{with } \nu = \lfloor i/2 \rfloor.$$

For $g \leqslant i \leqslant 2g$ the decomposition takes the form

$$R^i(X/S) = R^{(g,i-g)} \oplus R^{(g-1,i+1-g)} \oplus \cdots \oplus R^{(g-\nu,i+\nu-g)}\,, \quad \text{again with } \nu = \lfloor i/2 \rfloor.$$

Fourier duality exchanges $R^{(\lambda_1,\lambda_2)}(X/S)$ and $R^{(g-\lambda_2,g-\lambda_1)}(X^\dagger/S)$. By looking at cohomology we can see that in general all summands $R^{(\lambda_1,\lambda_2)}$ in the indicated range are non-zero.

(5.5) *Remark.* — If we drop the assumption that X is isogenous to a power of a simple abelian scheme over S, we may proceed as in (2.2). Choose an isogeny $h\colon X \to Y_1 \times \cdots \times Y_r$ such that each Y_ν is isogenous to a power of a simple abelian scheme. To each Y_ν we may apply (5.1). As h induces an isomorphism $R(X/S) \cong R(Y_1/S) \otimes \cdots \otimes R(Y_r/S)$, this gives us a refined decomposition of the Chow motive of X. We leave it to the reader to write out the details.

It is instructive to consider the case where X is isogenous to Y^r for some abelian scheme Y/S with $\mathrm{End}(Y/S) = \mathbb{Z}$. In this case, taking $Y_1 = \cdots = Y_r = Y$ gives back the decomposition of (2.2), which, in general, is finer than the decomposition of $R(X/S)$ we obtain by applying (5.1) to X itself. However, the finer decomposition in (2.2) does not give information on how $\mathrm{GL}_r(\mathbb{Q})$ acts; it only takes into account the action of the diagonal subgroup $\mathbb{Q}^* \times \cdots \times \mathbb{Q}^*$ (r factors).

(5.6) *Remark.* — There is another, perhaps more elementary, way to obtain a motivic decomposition of $R(X/S)$, which coincides with the decomposition of (5.1) if $D = K$ but which in general is coarser. For this we need to work in the category $\mathbf{M}^0(S;\tilde{K})$ of relative Chow motives with coefficients in $\tilde{K}$. Write $R^i(X/S;\tilde{K})$ for the image of $R^i(X/S)$ under the natural functor $\mathbf{M}^0(S) \to \mathbf{M}^0(S;\tilde{K})$.

Let $D_{\tilde{K}} = D \otimes_{\mathbb{Q}} \tilde{K}$. Then $D_{\tilde{K}} = \prod_{\sigma \in \Sigma(K)} D_\sigma$, where $D_\sigma = D \otimes_{K,\sigma} \tilde{K}$. Let $1 = \sum e_\sigma$ be the corresponding decomposition of $1 \in D_{\tilde{K}}$ as a sum of idempotents. By (2.5) we have an algebra homomorphism $r_{\tilde{K}}\colon D_{\tilde{K}}^{\mathrm{opp}} \to \mathrm{End}_{\mathbf{M}^0(S;\tilde{K})}\big(R^1(X/S;\tilde{K})\big)$. This gives a decomposition $R^1(X/S;\tilde{K}) = \oplus_{\sigma \in \Sigma(K)} R_\sigma$, where R_σ is the submotive of $R^1(X/S;\tilde{K})$ cut out by the idempotent $r_{\tilde{K}}(e_\sigma)$.

Let $\boldsymbol{J} = (\mathbb{Z}_{\geqslant 0})^{\Sigma(K)}$, and for $i \geqslant 0$ define a subset $\boldsymbol{J}(i) \subset \boldsymbol{J}$ by

$$\boldsymbol{J}(i) = \bigl\{\boldsymbol{j}\colon \Sigma(K) \to \mathbb{Z}_{\geqslant 0} \bigm| |\boldsymbol{j}| = i\bigr\}\,,$$

where $|\boldsymbol{j}| = \sum_{\sigma\in\Sigma(K)} \boldsymbol{j}(\sigma)$. Taking exterior powers and using Künnemann's isomorphism $\wedge^i R^1(X/S) \xrightarrow{\sim} R^i(X/S)$, we obtain decompositions

$$R^i(X/S;\tilde{K}) = \bigoplus_{\boldsymbol{j}\in\boldsymbol{J}(i)} R^{(\boldsymbol{j})}(X/S;\tilde{K}) \qquad \text{such that} \quad R^{(\boldsymbol{j})}(X/S;\tilde{K}) \cong \bigotimes_{\sigma\in\Sigma(K)} \Bigl(\wedge^{\boldsymbol{j}(\sigma)} R_\sigma\Bigr)\,.$$

(The calculation of the exterior powers works as expected; cf. [3], Section 1.) Fixing $i \geqslant 0$, let $1 = \sum_{\boldsymbol{j}\in\boldsymbol{J}(i)} \epsilon_{\boldsymbol{j}}$ be the corresponding decomposition of $1 \in \mathrm{End}_{\mathbf{M}^0(S;\tilde{K})}\bigl(R^i(X/S;\tilde{K})\bigr)$ as a sum of idempotents. The Galois group Γ acts on $\boldsymbol{J}(i)$ and on the endomorphism algebra of the motive $R^i(X/S;\tilde{K})$. If $\gamma\in\Gamma$ sends $\boldsymbol{j}\in\boldsymbol{J}(i)$ to $\boldsymbol{j}'$ then ${}^\gamma\epsilon_{\boldsymbol{j}} = \epsilon_{\boldsymbol{j}'}$. Hence if β is a Γ-orbit in $\boldsymbol{J}(i)$, the sum $\sum_{\boldsymbol{j}\in\beta} \epsilon_{\boldsymbol{j}}$ is an idempotent in $\mathrm{End}_{\mathbf{M}^0(S)}\bigl(R^i(X/S)\bigr)$. This gives us a decomposition

$$R^i(X/S) = \bigoplus_{\beta\in\boldsymbol{J}(i)/\Gamma} R^{(\beta)}(X/S)$$

in $\mathbf{M}^0(S)$ such that $R^{(\beta)}(X/S;\tilde{K}) = \oplus_{\boldsymbol{j}\in\beta}\, R^{(\boldsymbol{j})}(X/S;\tilde{K})$.

To describe the relation with (5.1), consider the map $v\colon \mathbb{X}^{\mathrm{adm}}/\Gamma \to \boldsymbol{J}/\Gamma$ that sends the Γ-orbit of $\boldsymbol{\lambda}\in\mathbb{X}^{\mathrm{adm}}$ to the Γ-orbit of the function $\sigma\mapsto|\boldsymbol{\lambda}(\sigma)|$. By analyzing how the groups $D_\sigma^{\mathrm{opp},*}$ act, we find that $R^{(\beta)}(X/S) = \oplus\, R^{(\alpha)}(X/S)$, where the sum runs over the classes $\alpha\in\mathbb{X}^{\mathrm{adm}}/\Gamma$ such that $v(\alpha)=\beta$. In particular, $R^{(\beta)}$ can only be non-zero if $|\boldsymbol{j}(\sigma)| \leqslant 2g/n$ for all $\boldsymbol{j}\in\beta$ and $\sigma\in\Sigma(K)$; hence $\wedge^j R_\sigma = 0$ for $j > 2g/n$.

References

[1] A. Beauville, *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne.* Algebraic geometry (Tokyo/Kyoto, 1982), 238–260. Lecture Notes in Math. 1016, Springer, Berlin, 1983.

[2] A. Beauville, *Sur l'anneau de Chow d'une variété abélienne.* Math. Ann. 273 (1986), 647–651.

[3] P. Deligne, *Catégories tensorielles.* Mosc. Math. J. 2 (2002), 227–248.

[4] C. Deninger, J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform.* J. reine angew. Math. 422 (1991), 201–219.

[5] W. Fulton, J. Harris, *Representation theory. A first course.* Graduate Texts in Mathematics 129. Springer-Verlag, New York, 1991.

[6] S. Kimura, *Correspondences to abelian varieties. I.* Duke Math. J. 73 (1994), 583–591.

[7] K. Künnemann, *On the Chow motive of an abelian scheme.* Motives (Seattle, WA, 1991), Part 1, 189–205. Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, RI, 1994.

[8] D. Mumford, *Abelian varieties.* Tata Institute of Fundamental Research Studies in Math. 5. Oxford University Press, Oxford, 1970.

[9] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque.* J. reine angew. Math. 247 (1971), 196–220.

[10] T. Wedhorn, *Ordinariness in good reductions of Shimura varieties of PEL-type.* Ann. scient. Éc. Norm. Sup. (4) 32 (1999), 575–618.

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